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# THE GEOMETRY OF LAGRANGE AND HAMILTON SPACES

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**Abstract** This paper is an overview of the main properties of Lagrange and Hamilton spaces.
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## 1. INTRODUCTION

The geometry of Lagrange spaces, introduced and studied by R.Miron and his collaborators was extensively examined in the last 25 years by geometers and physicists from: Canada, Germany, Hungary, Italy, Japan, Romania, Russia, Jugoslavia, India, Egypt and USA. Many international conferences were devoted to debate this subject, proceedings and monographs were published, especially by Kluwer Academic Publishers. A large area of applicability of this geometry is suggested by the connection to Biology, Mechanics and Physics and also by its general setting as a generalization of Finsler and Riemannian geometries.

The concept of Hamilton space, introduced by R.Miron in 1987, was intensively studied in Romania, Japan and Canada as a geometric theory of the Hamiltonian function, a fundamental entity in Mechanics and Physics. The classical Legendre's duality makes possible a natural connection between Lagrange and Hamilton spaces. It reveals new concepts and geometrical objects of Hamilton spaces that are dual to those which are similar in Lagrange spaces. Following this duality it has been introduced the Cartan spaces, as dual of Finsler spaces.

In Lagrange geometry, the following sequence is fundamental

$$\{\mathcal{R}^n\} \subset \{F^n\} \subset \{L^n\} \subset \{GL^n\}$$

where $\{\mathcal{R}^n\}$ - Riemann spaces, $\{F^n\}$ - Finsler spaces, $\{L^n\}$ - Lagrange spaces, $\{GL^n\}$ - generalized Lagrange spaces.

In Hamilton geometry, we have

$$\{\mathcal{R}^{*n}\} \subset \{\mathcal{C}^n\} \subset \{H^n\} \subset \{GH^n\}$$

where $\{\mathcal{R}^{*n}\}$ - Riemann spaces (given by $g^{ij}$), $\{\mathcal{C}^n\}$ - Cartan spaces, $\{H^n\}$ - Hamilton spaces, $\{GH^n\}$ - generalized Hamilton spaces.

These sequences are related by Legendre diffeomorphism.

The geometry of Lagrange and Hamilton spaces is the geometrical theory of these two sequences. The applications in Mechanics, Relativity, Relativistic Optics, Variational Calculus, Optimal Control or Biology use the previous sequences.

There is a natural extension of this geometry given to the higher-order Lagrange and Hamilton geometry. Considering the higher-order acceleration bundle $(T^kM,\ \pi^k,\ M)$ we introduced the notion of Lagrange space of order $k$, as a pair $L^{(k)n} = (M,\ L)$, $M$ being a real $n$-dimensional manifold and $L : T^kM \to \mathbb{R}$ being a regular Lagrangian of order $k$.

**Remark.** In the first my visit at the University of Alberta in Edmonton (1995), P. L. Antonelli questioned me: "Have you examples of the regular Lagrangians of order $k \geq 3$?" I was surprised to constat that at that time there were no such kind of examples. This difficulty came from the fact that the old problem, formulated by Bianchi and Bompiani, of the prolongation of the Riemannian structure $g$, defined on the base manifold $M$ to the bundle $T^kM$ has not been solved yet.

So I was forced to solve this problem in order to apply it in the construction of the geometry of the spaces $L^{(k)n}$, obtaining the following sequence

$$\{\mathcal{R}^{(k)n}\} \subset \{F^{(k)n}\} \subset \{L^{(k)n}\} \subset \{GL^{(k)n}\}$$

where $\{\mathcal{R}^{(k)n}\}$ - Riemann spaces of order $k$, $\{F^{(k)n}\}$ - Finsler spaces of order $k$, $\{L^{(k)n}\}$ - Lagrange spaces of order $k$, $\{GL^{(k)n}\}$ - generalized Lagrange spaces of order $k$.

Kluwer Academic Publishers published this geometry in FTPH, 1997, nr.82.

The dual notion of the bundle $(T^kM,\ \pi^k,\ M)$ is defined by $(T^{*k}M,\ \pi^{*k},\ M)$ where

$$T^{*k}M = T^{k-1}M \times_M T^*M,\ \ \pi^{*k} = \pi^{k-1} \times \pi^*,$$

$(T^*M,\ \pi^*,\ M)$ being the cotangent bundle of the manifold $M$.

A Hamiltonian space of order $k$ is a pair $H^{(k)n} = (M,\ H)$ where $H : T^{*k}M \to \mathbb{R}$ is a regular Hamiltonian. One obtains a sequence similar with the previous and an extension of the Legendre transformation.

The geometry of the spaces $H^{(k)n}$ will be published in a book, this year, by Kluwer Academic Publishers FTPH.

**Geometry of** $TM$ (few notions and notations):

The change of coordinates on tangent bundle $TM$

$$\begin{cases} \tilde{x}^i = \tilde{x}^i(x^j), \quad rank \left\| \frac{\partial \tilde{x}^i}{\partial x^j} \right\| = n \\ \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j \end{cases} \tag{1}$$

where $\tilde{x}^i = \tilde{x}^i(x^j)$ is a change of coordinate on $M$

$V$ - vertical distribution $\left(\frac{\partial}{\partial y^i}\right)_{i=\overline{1,n}}$
$\Gamma$ - Liouville vector field
Tangent structure $J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \;\; J\left(\frac{\partial}{\partial y^i}\right) = 0$
Semispray $S \in \chi(TM), \;\; JS = \Gamma$

$$S = y^i \partial x^i - 2G^i(x, y)\partial y^i.$$

Integral curves

$$y^i = \frac{dx^i}{dt}, \;\; \frac{dy^i}{dt} + 2G(x, y) = 0$$

$G^i$ - coefficients, $\widetilde{TM} = TM \setminus \{0\}$.

## 2. LAGRANGE SPACES

Let $M$ be a differentiable manifold and $(TM, \; \pi, \; M)$ its tangent bundle. A *differentiable Lagrangian* is a mapping $L : TM \to \mathbb{R}$ which is smooth on $\widetilde{TM}$ and continuous on the null section $0 : M \to TM$ of the projection $\pi : TM \to M$.

**Definition 2.1.** *A Lagrange space is a pair $L^n = (M, L)$ satisfying the axioms:*
*1) $L : TM \to \mathbb{R}$ is differentiable of class $C^\infty$ on $\widetilde{TM}$ and continuous on the null section of projection $\pi : TM \to M$,*
*2) the Hessian of L is nonsingular,*
*a) $g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}$,*
*b) $rank\|g_{ij}\| = n$,*
*3) the tensor field $g_{ij}$ has a constant signature on $\widetilde{TM}$.*

Let $c : t \in [0, 1] \to (x^i(t)) \in \mathcal{U} \subset M$ be a curve. The action integral of the Lagrangian $L$ on $c$ is given by

$$I(c) = \int_0^1 L(x(t), \frac{dx}{dt})\, dt.$$

Then, the Euler-Lagrange Equation is

$$E_i(L) := \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^i} = 0, \;\; y^i = \frac{dx^i}{dt}\, dt,$$

while the energy of $L$ is

$$E_L = y^i \frac{\partial L}{\partial y^i} - L.$$

Law of conservation: *For any differentiable Lagrangian L(x,y) the energy $E_L$ is conserved along every solution curve c of the Euler-Lagrange equations*

$$E_i(L) = 0, \quad \frac{dx^i}{dt} = y^i.$$

Nöther theorem: *For any infinitesimal symmetry on $M \times \mathbb{R}$ of the Lagrangian L(x,y) and for any smooth function $\Phi(x)$ the following function*

$$\mathcal{F}(L, \Phi) = V^i \frac{\partial L}{\partial y^i} - \tau E_L - \Phi(x)$$

*is conserved on every solution curve c of the Euler-Lagrange equations $E_i(L) = 0$, $\frac{dx^i}{dt} = y^i$.*

For a Lagrange space $L^n$ the Euler-Lagrange equation has the form

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0, \quad y^i = \frac{dx^i}{dt}$$

where

$$2G^i(x, y) = \frac{1}{2} g^{ij} \left( \frac{\partial^2 L}{\partial y^j \partial} x^h y^h - \frac{\partial L}{\partial x^i} \right)$$

are the coefficients of the canonical semispray $S$.

The geometry of $L^n$ can be constructed only by means of geometric objects $L,\ g^{ij},\ S$.

The nonlinear connection $N$:

$$T_u = N_u \oplus V_n, \quad \forall u \in TM$$

has the adapted basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}.$$

It is canonical determined by $S$ because:

$$N_j^i = \frac{\partial G^i}{\partial y^j}.$$

If $D$ is an $N$-linear connection with the coefficients $D\Gamma(N) = \left(L^i_{jk}, C^i_{jk}\right)$ the h- and v- covariant derivation of the fundamental tensor are:

$$\begin{aligned} g_{ij|k} &= \tfrac{\delta g_{ij}}{\delta x^k} - g_{sj} L^s_{ik} - g_{is} L^s_{jk} \\ g_{ij}\,|_k &= \tfrac{\partial g_{ij}}{\partial y^k} - g_{sj} C^s_{ik} - g_{is} C^s_{jk}. \end{aligned}$$

The coefficients $D\Gamma(N) = \left(L^i_{jk}, C^i_{jk}\right)$ of $D$ are expressed by the following generalized Christoffel symbols:

$$\begin{aligned} L^i_{jk} &= \tfrac{1}{2} g^{is} \left( \tfrac{\delta g_{sk}}{\delta x^j} + \tfrac{\delta g_{js}}{\delta x^k} - \tfrac{\delta g_{jk}}{\delta x^s} \right) \\ C^i_{jk} &= \tfrac{1}{2} g^{is} \left( \tfrac{\partial g_{sk}}{\partial x^j} + \tfrac{\partial g_{js}}{\partial x^k} - \tfrac{\partial g_{jk}}{\partial x^s} \right). \end{aligned}$$

An N-connection $D\Gamma(N)$ is metrical if

$$g_{ij|k} = 0, \;\; g_{ij}\,|_k \; = 0.$$

Dual adapted basis is $(dx^i, \; \delta y^i)$ with $\delta y^i = dy^i + N^i_j dx^j$.
Then, the 1-forms connection of $D\Gamma(N)$ are

$$\omega^i_j = L^i_{jk} dx^k + C^i_{jk} \delta y^k$$

while the structure equations are:

$$\begin{aligned} d(dx^i) - dx^k \wedge \omega^i_k &= -\overset{(0)}{\Omega}{}^i \\ d(\delta y^i) - \delta y^k \wedge \omega^i_k &= -\overset{(1)}{\Omega}{}^i \\ d\omega^i_j - \omega^k_j \wedge \omega^i_k &= -\Omega^i_j. \end{aligned}$$

Here the 2-forms of torsion $\overset{(0)}{\Omega}{}^i$ and $\overset{(1)}{\Omega}{}^i$ are:

$$\begin{aligned} \overset{(0)}{\Omega}{}^i &= C^i_{jk} dx^j \wedge \delta y^k \\ \overset{(1)}{\Omega}{}^i &= \tfrac{1}{2} R^i{}_{jk} dx^j \wedge dx^k + P^i{}_{jk} dx^j \wedge \delta y^k \end{aligned}$$

where

$$R^h{}_{ij} = \frac{\delta N^h_i}{\delta x^j} - \frac{\delta N^h_j}{\delta x^i}, \;\; P^i{}_{jk} = \frac{\partial N^i_j}{\partial y^k} - L^i{}_{kj}.$$

The 2-forms of curvature $\Omega^i_j$ are given by

$$\Omega^i_j = \frac{1}{2} R_j{}^i{}_{ks} dx^k \wedge dx^s + P_j{}^i{}_{ks} dx^k \wedge \delta y^s + \frac{1}{2} S_j{}^i{}_{ks} \delta y^k \wedge \delta y^s$$

where

$$\begin{aligned} R_h{}^i{}_{jk} &= \tfrac{\delta L^i_{hj}}{\delta x^k} - \tfrac{\delta L^i_{hk}}{\delta x^j} + L^s_{hj} L^i_{sk} - L^s_{hk} L^i_{sj} + C^i_{hs} R^s_{jk} \\ P_h{}^i{}_{jk} &= \tfrac{\partial L^i_{hj}}{\partial y^k} - C^i_{hk|j} + C^i_{hs} P^s_{jk} \\ S_h{}^i{}_{jk} &= \tfrac{\partial C^i_{hj}}{\partial y^k} - \tfrac{\partial C^i_{hk}}{\partial y^j} + C^s_{hj} C^i_{sk} - C^s_{hk} C^i_{sj}. \end{aligned}$$

Ricci tensors are defined by:

$$R_{ij} = R^k_{ijk}, \;\; \overset{1}{P}_{ij} = P^k_{ijk}, \;\; \overset{2}{P}_{ij} = P^k_{ikj}, \;\; S_{ij} = S^k_{ijk}$$

**Theorem 2.1.** *In the adapted frame the Einstein equations are:*

$$\begin{array}{ll} R_{ij} - \frac{1}{2}Rg_{ij} = \kappa \overset{1}{T}_{ij} & S_{ij} - \frac{1}{2}S g_{ij} = \kappa \overset{2}{T}_{ij} \\ \overset{1}{P}_{ij} = \kappa \overset{3}{T}_{ij} & \overset{2}{P}_{ij} = \kappa \overset{4}{T}_{ij}. \end{array}$$

*The law of conservation is:*

$$\left\{ R^i{}_j - \frac{1}{2}R\delta^i_j \right\}_{|i} + \overset{1}{P}{}^i_{j|i} = 0.$$

**Electromagnetic field**

Let us consider a Lagrange space $L^n = (M, L)$ endowed with the canonical nonlinear connection $N$ and with the canonical metrical N-connection $C\Gamma(N) = (L^i_{jk}, C^i_{jk})$. The covariant deflection tensors $D_{ij}$ and $d_{ij}$ are given by $D_{ij} = g_{is}D^s{}_j,\ d_{ij} = g_{is}d^s{}_j$

The d-tensor fields

$$F_{ij} = \frac{1}{2}(D_{ij} - D_{ji}), \quad f_{ij} = \frac{1}{2}(d_{ij} - d_{ji})$$

are the h- and v-electromagnetic tensor of the Lagrange space $L^n = (M, L)$. The following generalized Maxwell equations hold:

$$\begin{aligned} & f_{ij} = 0, \\ & F_{ij|k} + F_{jk|i} + F_{ki|j} = -\sum_{(ijk)} C_{i0r}R^r{}_{jk} \\ & F_{ij}\big|_k + F_{jk}\big|_i + F_{ki}\big|_j = 0 \end{aligned}$$

where $C_{i0s} = C_{ijs}y^j$ and $\sum\limits_{(ijk)}$ means cyclic sum.

**Almost Kählerian model of space $L^n$**

A Lagrange space $L^n = (M, L)$ can be thought as an almost Kähler space on the manifold $\widetilde{TM} = M \setminus \{0\}$. The canonical nonlinear connection $N$ determines an almost complex structure $F(\widetilde{TM})$ where

$$F = \frac{\delta}{\delta x^i} \otimes \delta y^i - \frac{\partial}{\partial y^i} \otimes dx^i.$$

The lift of the fundamental tensor $g_{ij}$ of $L^n$ with respect to $N$ is defined by:

$$G = g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j.$$

Then $(G, F)$ is an almost Kählerian space with the almost symplectic structure associated to $(G,\ F)$ given by

$$\theta = g_{ij}\delta y^i \wedge d^j.$$

**Theorem 2.2.** *The canonical connection $D$ with coefficients $C\Gamma(N) = (L^i_{jk}, C^i_{jk})$ of the Lagrange space $L^n$ is an almost Kählerian connection, i.e.*

$$DG = 0,\ DF = 0.$$

**Finsler Spaces**

**Theorem 2.3.** *A Finsler space is a pair $F^n = (M, F(x,y))$, where $M$ is a real $n$-dimensional differentiable manifold and $F : TM \to \mathbb{R}$ is a mapping which satisfies the following axioms:*

*1) $F$ is a differentiable function on $\widetilde{TM}$ and is only continuous on the null section of the projection $\pi : TM \to M$,*

*2) $F$ is a positive function,*

*3) $F$ is positively 1-homogeneous with respect to the variables $y^i$*

*4) the Hessian of $F^2$ with the elements*

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial F^2}{\partial y^i \partial y^j}$$

*is positively defined on $\widetilde{TM}$.*

Any Finsler space $F^n = (M, F(x,y))$ is a Lagrange space $L^n_F = (M, F^2(x,y))$. Then

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$$

is a spray which allows to define a Cartan nonlinear connection. This Cartan connection helps to get a canonical Cartan metrical connection.

An example of a special Lagrange space derived from a Finsler $F^n = (M, F(x,y))$ is defined by means of function $L(x,y) = F^2(x,y) + b_i(x)y^i$; this Lagrange space is called the almost Finsler Lagrange Space (shortly AFL-space). These spaces generalize the Lagrange space from Electrodynamics.

**Generalized Lagrange spaces**

**Definition 2.2.** *A generalized Lagrange space is a pair $GL^n = (M, g_{ij}(x,y))$, where $g_{ij}(x,y)$ is a d-tensor field on the manifold $\widetilde{TM}$, of type (0,2), symmetric, of rank $n$ and having a constant signature on $\widetilde{TM}$.*

This notion was introduced by author.

For a generalized Lagrange space $GL^n$ an important problem is to determine a nonlinear connection obtained from the fundamental tensor $g_{ij}(x,y)$. To this end we consider the absolute energy $\varepsilon(x,y) = g_{ij}(x,y)j^i y^j$. In certain conditions $\varepsilon(x,y)$ is a Lagrangian for which we get easily its Euler-Lagrange equations. These equations

determine a semispray which allows to get the nonlinear connection $N$ and to study the geometry of pair $(GL^n, N)$ by the methods of the geometry of Lagrange space $L^n$.

The model of $GL^n$ is not almost Kählerian space.

**Applications in Theoretical Physics**

The pair $GL^n = (M, g_{ij}(x, y))$

$$g_{ij}(x, y) = mc\gamma_{ij}(x) + \left(1 - \frac{1}{n^2(x, y)}\right) y_i y_j$$

where $\gamma_{ij}(x)$ is a pseudo-Riemannian metric and $n(x, y) > 1$ is a smooth function, give us a generalized Lagrange space $GL^n$ which is not reducible to a Lagrange space. This metric has been called by R.Beil the Miron metric from Relativistic Optics.

**Hamilton Spaces**

**Definition 2.3.** *A Hamilton space is a pair $H^n = (M, H(x, p))$, where $H(x, p)$ is a real function on the momentum space $T^*(M)$ having the following properties:*

*1) $H : T^*M \to \mathbb{R}$ is differentiable on $\widetilde{T^*M} = T^*M \setminus \{0\}$ and only continuous on the null section of projection $\pi_*$*

*2) the Hessian of H, with the elements*

$$g^{ij} = \frac{1}{2}\dot{\partial}^i\dot{\partial}^j H$$

*is nonsingular, i.e. $det(g^{ij}) \neq 0$ on $\widetilde{T^*M}$*

*3) the 2-form $g^{ij}(x, p)\eta_i\eta_j$ has a constant signature on $\widetilde{T^*M}$.*

Recall that on $T^*(M)$ there exists a globally defined the Liouville 1-form

$$\omega = p_i dx^i$$

and a natural symplectic structure $\theta = d\omega = dp_i \wedge dx^i$.

There exists a unique vector field $X_H$ on $T^*M$ such that $i_{X_H}\theta = -dH$.

The integral curve of $X_H$ are given by the Hamilton-Jacobi equations

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}.$$

**The variational problem applied to the action integral of $H^n$**

If

$$c : t \in [0, 1] \to (x^i(t), p_i(t)) \in T^*M$$

then the action integral:

$$I(c) = \int_0^1 [p_i \frac{dx^i}{dt} - \frac{1}{2}H(x(t), p(t))]dt$$

gives us the Hamilton-Jacobi equations.

There is a canonical nonlinear connection and a N-linear connection which depend on the Hamiltonian $H$ only.

The geometry of $H^n$ can be lifted to the total space of cotangent bundle. One obtains an almost Kählerian space.

**Application of Hamilton space to electrodynamics**

A Hamiltonian mechanical system is a triple $\Sigma_H = (M, H(x,p), Fe(x,p))$ where $H^n = (M, H(x,p))$ is a Hamilton space and $Fe(x,p)) = F_i(x,p)\dot{\partial}^i$ is a given vertical vector field on the momenta space $T^*M$. $Fe$ is called *the external forces field*.

The evolution equations of the Hamiltonian mechanical system $\Sigma_H$ are the following Hamilton equations:

$$\frac{dx^i}{dt} - \dot{\partial}^i\mathcal{H} = 0, \quad \frac{dp^i}{dt} + \partial_i\mathcal{H} = \frac{1}{2}F_i(x,p), \quad \mathcal{H} = \frac{1}{2}H.$$

Consider $H^n = (M, H(x,p))$ the Hamilton space of electrodynamics

$$H(x,p) = \frac{1}{mc}\gamma^{ij}(x)p_ip_j - \frac{2e}{m^2c}A^i(x)p_i + \frac{e^2}{m^3c}A^i(x)A_i(x)$$

and $Fe(x,p)) = F_i(x,p)\dot{\partial}^i$. Then $\Sigma_H$ is a Hamiltonian mechanical system determined only by $H^n$.

**Proposition 2.1.** *The fundamental function $H(x,p)$ of the Hamiltonian mechanical system $\Sigma_H$ is decreasing on the evolution curves of $\Sigma_H$, if and only if the external forces $Fe$ are dissipative.*

Legendre differentiation is a mapping locally defined by:

$$\mathcal{L} : L^n = (M, L(x,y)) \to H^n = (M, H(x,p)), \quad x^i = x^i, \quad p_i = \tfrac{1}{2}\tfrac{\partial L}{\partial y^i}$$
$$\varphi(x,y) = (x^i, \dot{\partial}_j\mathcal{L}(x,y));$$

it transforms fundamental object fields of $L^n$ into fundamental object fields of $H^n$.

**Cartan spaces**

**Definition 2.4.** *A Cartan space is a pair $C^n = (M, K(x,p))$ where $M$ is a real $n$-dimensional smooth manifold and $K : T^*M \to \mathbb{R}$ is a scalar function which satisfies the following axioms:*

*1) $K$ is a differentiable function on $\widetilde{T^*M}$ and only continuous on the null section of the projection $\pi^* : T^*M \to M$,*

*2) $K$ is a positive function on the manifold $T^*M$,*

*3) $K$ is positive 1-homogeneous with respect to the momenta $p_i$,*

*4) the Hessian of $K^2$ with the elements*

$$g^{ij}(x,p) = \frac{1}{2}\dot{\partial}^i\dot{\partial}^j K^2$$

*is positively defined on $\widetilde{T^*M}$.*

A particular class of Hamilton spaces is furnished by Cartan spaces. Any Cartan space is a pair $H^n = (M, H(x,p))$ for which the fundamental function $H$ is 2-homogeneous with respect to momenta $p_i$.

The Cartan spaces are dual of Finsler spaces, via Legendre transformation. This is the reason why the geometry of Cartan spaces has the same importance, symmetry and beauty as Finsler spaces.

**Concluding:** The geometry of Lagrange and Hamilton spaces can be based and developed on the mechanical principles.

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